Oct. 24, 2013

EE 301 - BIBO Stability of LTI Systems

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During the lecture hour, we have said that if the impulse response of a LTI system is absolutely summable[1], the system is stable (BIBO stable).

This statement is not mathematically as accurate as you may want it to be. Let's study the same statement a little bit more formally.

i. **Sufficiency of Absolute Summability:** $(\rightarrow)$
We first show the following:

$$\text{(Absolute Summable Impulse Response)} \rightarrow \text{(BIBO Stability)}$$

This statement says that if it is granted that the impulse response is absolutely summable; then system is BIBO stable[2]. The arrow "$\rightarrow$" indicates the logical implication of left side over the right side. Such logic statements are read in plain english as (Absolute Summable Impulse Response) implies (BIBO Stability).

Let's try to show this logical implication for the discrete-time systems. (The development for the continuous-time systems is almost identical). To do that we assume the left side of the implication is true and try to reach the one on the right.

We assume that:

1. There exists a real number $\mathrm{M_x}$ such that $|\mathrm{x[n]}| \le \mathrm{M_x}\ \ \forall n$ (Bounded input)
2. There exists a real number $\mathrm{M_h}$ such that $\sum_{n=-\infty}^{\infty} |h[n]| \le M_h$ (Absolutely summable impulse response)

Since $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, we have the following

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]| \le M_x \sum_{k=-\infty}^{\infty} |h[k]| \le M_x M_h$$

The last line shows that $|y[n]| \le M_x M_h$; hence $y[n]$ remains bounded (for every bounded input $\mathrm{x[n]}$) which is the result that we are trying to show.

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[1] Absolutely summability is $\sum_{n=-\infty}^{\infty} |h[n]|$ to be finite number, ($\sum_{n=-\infty}^{\infty} |h[n]| < M$ for a real valued M) for the discrete time systems and it is $\int_{-\infty}^{\infty} |h(t)| dt < M$ for continuous time systems.

[2] BIBO Stability is bounded input $\mathrm{x[n]}$ results in bounded output $y[n]$. Similarly for continous-time systems.

ii. **Necessity of Absolute Summability:** $(\leftarrow)$

Next, we show the necessity part. In other words, we try to show that *every* BIBO stable LTI system has an absolutely summable impulse response. This is reverse argument stating that BIBO stability implies absolute summability:

(Absolute Summable Impulse Response) $\leftarrow$ (BIBO Stability)

We assume BIBO stability, that is

1. There exists a real number M_x such that $|x[n]| \le M_x \ \forall n$ (Bounded input)
2. The system produces output $y[n]$ to the input $x[n]$
3. There exists a real number M_y such that $|y[n]| \le M_y \ \forall n$ (Bounded output)

and try to reach the condition $\sum_{n=-\infty}^{\infty} |h[n]| \le M_h$.

Since $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$, we have $|y[0]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[-k]\right| \le M_y$. Then, by choosing $x[-k] = M_x sign(h[k])$ (please note that this choice satisfies the assumption of $|x[n]| \le M_x$), $|y[0]|$ becomes

$$|y[0]| = \left| M_x \sum_{k=-\infty}^{\infty} |h[k]| \right| \le M_y$$

Since every term in the summation is positive, we can write the same result as:

$$|y[0]| = M_x \sum_{k=-\infty}^{\infty} |h[k]| \le M_y$$

The last line shows that $\sum_{k=-\infty}^{\infty} |h[k]| \le \frac{M_y}{M_x}$, i.e. h[k] is absolutely summable.

As a summary, we have shown that absolute summability of the impulse response of an LTI system is equivalent to the BIBO stability. In other words, these two results imply each other and they are equivalent:

(Absolute Summable Impulse Response) $\leftrightarrow$ (BIBO Stability)